[6]Abujelala, M.T. and Lilley, D.G., "Confined Swirling Flow Predictions," AIAA Paper No. 83-0316, Jan. 1983.
[7]Habib, M.A. and Whitelaw, J.H., "Velocity Characteristics of Confined Coaxial Jets with and without Swirl," *Journal of Fluids Engineering,* Vol. 102, March 1980, pp. 47-53.
[8]Gibson, M.M., "An Algebraic Stress and Heat-Flux Model for Turbulent Shear Flow with Streamline Curvature," *International Journal of Heat and Mass Transfer,* Vol. 21, Dec. 1978, pp. 1609-1617.

Effect of Thermal Gradient on Frequencies of a Wedge-Shaped Rotating Beam

J.S. Tomar* and Rita Jain†
University of Roorkee, Roorkee, India

Introduction

SUFFICIENT work is available on vibrations of rotating beams[1-5] but none of them have considered the thermal effect on frequencies of bending vibrations. It is well known[6] that in the presence of a constant thermal gradient the elastic coefficients of homogeneous materials become functions of the space variables. Fanconneau and Marangoni[7] have investigated the effect of the nonhomogeneity caused by a thermal gradient on the natural frequencies of simply supported plates of uniform thickness.

The analysis presented in this Note considers bending vibrations of a wedge-shaped beam that could represent a turbine blade of simple geometry. The beam is attached to a disk of radius r as indicated in Fig. 1; the disk rotates with angular velocity Ω. It is restricted to the analysis of pure bending. Furthermore, the beam is subjected to one-dimensional temperature distribution along the length. Because the beam is wedge shaped, the area of cross section varies linearly. Consequently, $I(x)$, $b(x)$, and $A(x)$ are functions of x.

Analysis and Equation of Motion

The governing differential equation of transverse motion of a rotating beam of variable cross section, according to Schilhansl,[1] is

$$\frac{\partial^2}{\partial x^2}\left[EI(x)\frac{\partial^2 V}{\partial x^2}\right]+A(x)\rho\frac{\partial^2 V}{\partial t^2}=\frac{\partial^2 M}{\partial x^2} \tag{1}$$

where, using $\delta=[1-A(L)/A(0)]$

$$A(x)=A(0)\left(1-\delta\frac{x}{L}\right)$$

$$I(x)=\frac{A(0)}{12}\left[b^2(0)\left(1-\delta\frac{x}{L}\right)^3+\left(1-\delta\frac{x}{L}\right)t_l^2\right]$$

and

$$\frac{\partial^2 M}{\partial x^2}=\rho A(0)\Omega^2\left[\left\langle r(L-x)+\left(1-\delta\frac{r}{L}\right)\frac{L^2-x^2}{2}-\delta\frac{(L^3-x^3)}{3L}\right\rangle\frac{\partial^2 V}{\partial x^2}-\left\langle r+\left(1-\delta\frac{r}{L}\right)x-\delta\frac{x^2}{L}\right\rangle\frac{\partial V}{\partial x}+\left(1-\delta\frac{x}{L}\right)\sin^2\Psi V\right]$$

Equation (1), when put in terms of dimensionless variable $\xi=x/L$, takes the form

$$D\frac{\partial^4 V}{\partial\xi^4}+2D_{,\xi}\frac{\partial^3 V}{\partial\xi^3}+D_{,\xi\xi}\frac{\partial^2 V}{\partial\xi^2}+L^4\rho A(0)(1-\delta\xi)\frac{\partial^2 V}{\partial t^2}$$
$$=L^4\rho A(0)\Omega^2\left[\left\langle\frac{r}{L}(1-\xi)+\frac{(1-\xi^2)}{2}\left(1-\delta\frac{r}{L}\right)-\frac{\delta}{3}(1-\xi^3)\right\rangle\frac{\partial^2 V}{\partial\xi^2}-\left\langle\frac{r}{L}+\left(1-\delta\frac{r}{L}\right)\xi-\delta\xi^2\right\rangle\frac{\partial V}{\partial\xi}+(1-\delta\xi)\sin^2\Psi V\right] \tag{2}$$

where

$$D=\frac{EA(0)}{12}[b^2(0)(1-\delta\xi)^3+(1-\delta\xi)t_l^2] \tag{3}$$

The comma followed by a subscript denotes partial differentiation with respect to the variable.

It is assumed that the beam is subjected to a steady one-dimensional temperature distributed along the length, i.e., in the x direction

$$T=T_0(1-\xi) \tag{4}$$

where T denotes the temperature excess above the reference temperature at any point at a distance $\xi=x/L$ and T_0 denotes the temperature excess above the reference temperature at the end $x=L$ or $\xi=1$.

The temperature dependence of the modulus of elasticity for most of the engineering material is

$$E(T)=E_l(1-PT) \tag{5}$$

where E_l is the value of the modulus of reference temperature, i.e., at $T=0$ along the x direction.

Taking the temperature at the end of the beam as the reference temperature, i.e., at $\xi=1$, the modulus variation becomes

$$E(\xi)=E_l[1-\alpha(1-\xi)] \tag{6}$$

where

$$\alpha=PT_0 \qquad (0\leq\alpha\leq 1)$$

Substitution of Eq. (6) in Eq. (3) gives

$$D=E_l[1-\alpha(1-\xi)]\left\{\frac{A(0)}{12}[b^2(0)(1-\delta\xi)^3+t_l^2(1-\delta\xi)]\right\} \tag{7}$$

Determination of the Frequency Parameter

The solution of Eq. (2) can be assumed to be of the form

$$V(\xi,t)=A_l f(\xi)e^{i\omega t} \tag{8}$$

Received Jan. 13, 1983; revision received Aug. 3, 1983.

*Professor, Department of Mathematics.
†Research Fellow, Department of Mathematics.

where A_l is a constant. The function $f(\xi)$ satisfies the geometrical as well as dynamical boundary conditions of the beam which are as follows.

$$f=\frac{\mathrm{d}f}{\mathrm{d}\xi}=0 \qquad \text{at} \quad \xi=0$$

$$\frac{\mathrm{d}^2f}{\mathrm{d}\xi^2}=\frac{\mathrm{d}^3f}{\mathrm{d}\xi^3}=0 \qquad \text{at} \quad \xi=1 \tag{9}$$

Substitution of Eqs. (7) and (8) in Eq. (2) and using $\beta=E_l/12\rho L^2$, the governing differential equation becomes

$$\begin{aligned}
&\frac{1}{L^2}\langle[1-\alpha(1-\xi)][b^2(0)(1-\delta\xi)^3+t_l^2(1-\delta\xi)]\rangle\frac{\mathrm{d}^4f}{\mathrm{d}\xi^4}\\
&-\frac{1}{L^2}\langle(2\delta)[1-\alpha(1-\xi)][3b^2(0)(1-\delta\xi)^2+t_l^2]\\
&-2\alpha[b^2(0)(1-\delta\xi)^3+t_l^2(1-\delta\xi)]\rangle\frac{\mathrm{d}^3f}{\mathrm{d}\xi^3}\\
&+\frac{1}{L^2}\langle\delta^2[1-\alpha(1-\xi)][6b^2(0)(1-\delta\xi)]\\
&-2\alpha\delta[3b^2(0)(1-\delta\xi)^2+t_l^2]\rangle\frac{\mathrm{d}^2f}{\mathrm{d}\xi^2}\\
&-\frac{\Omega^2}{\beta}\left[\left\langle\frac{r}{L}(1-\xi)+\left(1-\delta\frac{r}{L}\right)\left(\frac{1-\xi^2}{2}\right)-\frac{\delta}{3}(1-\xi^3)\right\rangle\frac{\mathrm{d}^2f}{\mathrm{d}\xi^2}\right.\\
&\left.-\left\langle\frac{r}{L}+\left(1-\delta\frac{r}{L}\right)\xi-\delta\xi^2\right\rangle\frac{\mathrm{d}f}{\mathrm{d}\xi}+(1-\delta\xi)\sin^2\Psi f\right]\\
&=\lambda(1-\delta\xi)f
\end{aligned} \tag{10}$$

where

$$\lambda=12\omega^2L^2\rho/E_l \tag{11}$$

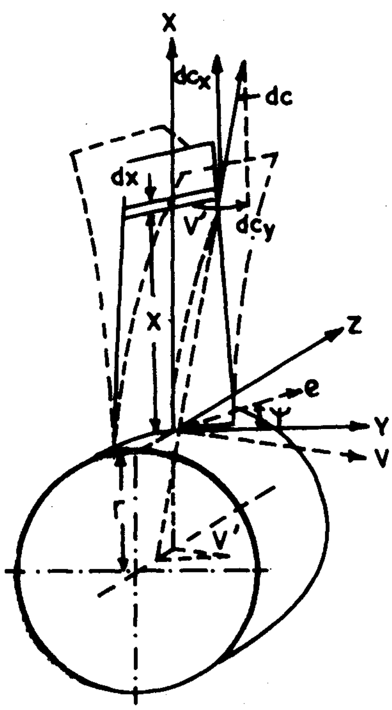

Fig. 1 Wedge-shaped beam attached to a circular disk.

is a frequency parameter. For an approximate determination of frequency parameter $\lambda, f(\xi)$ is chosen as the shape function for the different modes of bending vibrations of a uniform cantilever beam. These shape functions satisfy boundary conditions (9) and are

$$f_n(\xi)=\cosh\lambda_n\xi-\cos\lambda_n\xi-\sigma_n(\sinh\lambda_n\xi-\sin\lambda_n\xi) \tag{12}$$

where λ_n and σ_n are constants corresponding to the nth mode of vibrations. The values of these constants for the first three modes of vibrations are taken as given in Ref. 8. Equation (10) can be solved approximately for the frequency parameter λ by application of the method described in Ref. 9. We multiply Eq. (10) by $f_n(\xi)$ and integrate with respect to ξ from 0 to 1 to obtain the familiar Rayleigh quotient as

Table 1 Frequency vs angular speed for $\alpha=0$, $\delta=0$, and $r/L=0$

	$\gamma^{1/2}$		
$\bar{\Omega}$	Our values	Ref. 4, Table 2	Exact values Ref. 4, Table 2
0	3.51600	3.51602	3.51602
1	3.68171	3.68163	3.68165
2	4.13739	4.13710	4.13732
3	4.79736	4.79644	4.79728
4	5.58509	5.58316	5.58500
5	6.44964	6.44650	6.44954
6	7.36049	7.35614	7.36037
7	8.29977	8.29440	8.29964
8	9.25699	9.25085	9.25684
9	10.22586	10.21920	10.22570
10	11.20248	11.19560	11.20230
11	12.18449	12.17750	12.18430
12	13.17040	13.1634	13.17020

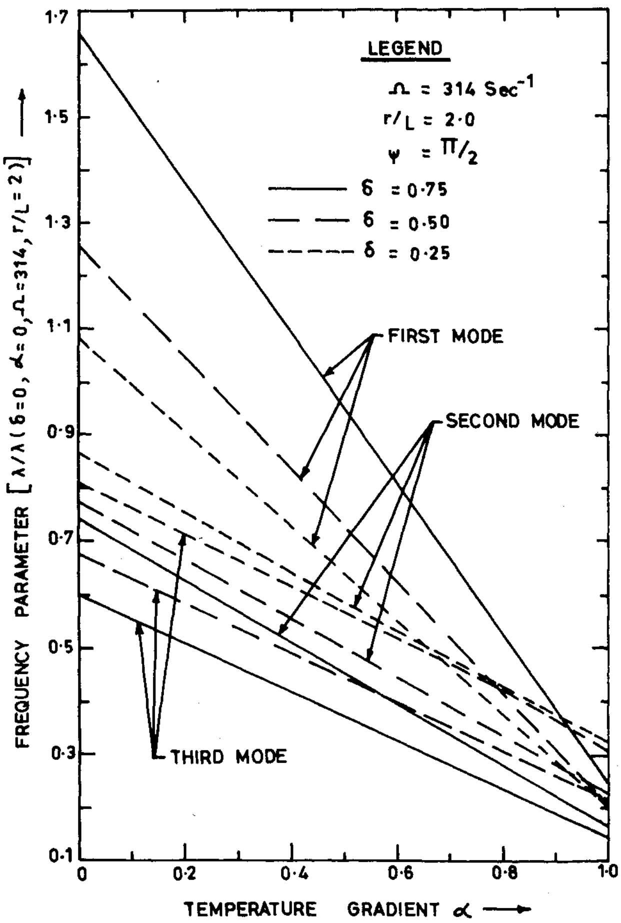

Fig. 2 Frequency parameter λ vs temperature gradient α for three different values of cross-sectional variation.

discussed by Collatz,[10] an equation of the form

$$a_1 - a_2 + a_3 + a_4 - a_5 - \lambda a_6 = 0 \tag{13}$$

where

$$a_1 = \frac{1}{L^2}\int_0^1 [1-\alpha(1-\xi)][b^2(0)(1-\delta\xi)^3 + t_1^2(1-\delta\xi)]\frac{d^4 f_n}{d\xi^4} f_n d\xi$$

$$a_2 = \frac{1}{L^2}\int_0^1 \langle 2\delta[1-\alpha(1-\xi)][3b^2(0)(1-\delta\xi)^2+t_1^2] - 2\alpha[b^2(0)(1-\delta\xi)^3 + t_1^2(1-\delta\xi)]\rangle \frac{d^3 f_n}{d\xi^3} f_n d\xi$$

$$a_3 = \frac{\delta}{L^2}\int_0^1 \langle \delta[1-\alpha(1-\xi)][6b^2(0)(1-\delta\xi)] - 2\alpha[3b^2(0)(1-\delta\xi)^2+t_1^2]\rangle \frac{d^2 f_n}{d\xi^2} f_n d\xi - \frac{\Omega^2}{\beta}\int_0^1 \left\langle \frac{r}{L}(1-\xi) + \left(1-\delta\frac{r}{L}\right)\left(\frac{1-\xi^2}{2}\right) - \frac{\delta}{3}(1-\xi^3)\right\rangle \frac{d^2 f_n}{d\xi^2} f_n d\xi$$

$$a_4 = \frac{\Omega^2}{\beta}\int_0^1 \left\langle \frac{r}{L} + \left(1-\delta\frac{r}{L}\right)\xi - \delta\xi^2\right\rangle \frac{d f_n}{d\xi} f_n d\xi$$

$$a_5 = \frac{\Omega^2}{\beta}\sin^2\Psi\int_0^1 (1-\delta\xi) f_n^2 d\xi$$

$$a_6 = \int_0^1 (1-\delta\xi) f_n^2 d\xi \tag{14}$$

The solution thus obtained would be an upper bound for the frequency parameter λ corresponding to the nth mode of vibration.

Results and Conclusions

The frequency parameter corresponding to first three modes of vibrations for various values of α and three different values of cross section variation (i.e., $\delta = 0.25$, $= 0.50$, $= 0.75$) are plotted in Fig. 2. Here the values of r/L and Ω are taken to be constant, equal to 2.0 and 314 s^{-1}, respectively, for a setting angle $\Psi = \pi/2$. The other physical constants are taken from Carnegie[11] and Rao.[12] It is observed that the frequencies decrease with an increase in α, while the frequencies increase with the increase in δ for the first mode and decrease for the second and third modes.

In order to justify the accuracy of the method employed, the results for the smallest eigenvalues have been compared with those of Ref. 4 after obtaining the frequencies from Eq. (13) using $\alpha = 0$, $\delta = 0$, $r/L = 0$ and computing

$$\gamma^{1/2} = \sqrt{[12\rho L^4/E_I(b^2(0)+t_1^2)](\omega^2+\Omega^2\sin^2\Psi)}$$

and

$$\bar{\Omega} = \Omega\sqrt{12\rho L^4/E_I[b^2(0)+t_1^2]}$$

Obviously, our results agree with the exact results quite well.

Further, the parameter $\bar{\Omega}$ corresponding to the rotational angular velocity $\Omega = 314$ s^{-1} is less than 1, the maximum error in frequency of vibrations of a rotating beam without thermal effect comes out to be 0.002% from Table 1. Also, as the frequency of vibrations of the rotating beam decreases with the increasing values of α, i.e., thermal gradient, the error will always be less than 0.002% when thermal effect is included.

References

[1] Schilhansl, M.J. and Previdence, R.I., "Bending Frequency of a Rotating Cantilever Beam," *Journal of Applied Mechanics*, Vol. 25, 1958, pp. 28-30.

[2] Carnegie, W., "The Application of the Variational Method to Derive the Equation of Motion of Vibrating Cantilever Blading under Rotation," *Bulletin of Mechanical Engineering Education*, Vol. 6, 1967, pp. 29-38.

[3] Tomar, J.S., "Coupled Vibrations of Slender Beam in a Centrifugal Force Field," *Proceedings of the National Institute of Science*, Vol. 35, 1969, pp. 629-639.

[4] Hodges, D.H., "An Approximate Formula for the Fundamental Frequency of a Uniform Rotating Beam Clamped Off the Axis of Rotation," *Journal of Sound and Vibration*, Vol. 77, No. 1, 1981, pp. 11-18.

[5] Rao, J.S. and Carnegie, W., "Non-linear Vibation of Rotating Cantilever Beams," *The Aeronautical Journal of the Royal Aeronautical Society*, Vol. 74, 1970, pp. 161-165.

[6] Hoff, N.J., *High Temperture Effect in Aircraft Structures*, Pergaman Press, New York, 1958.

[7] Fanconneau, G. and Marangoni, R.D., "Effect of a Thermal Gradient on the Natural Frequencies of a Rectangular Plate," *International Journal of Mechanical Sciences*, Vol. 12, 1970, pp. 113-122.

[8] Bishop, R.E.D. and Johnson, D.C., *The Mechanics of Vibration*, Cambridge University Press, London, 1962, p. 382.

[9] Fung, Y.C., *An Introduction to the Theory of Aeroelasticity*, John Wiley and Sons Inc., New York, 1965, p. 52.

[10] Collatz, L., *The Numerical Treatment of Differential Equations*, Springer-Verlag, Berlin, 1960, p. 228.

[11] Carnegie, W., "Vibrations of Pre-twisted Cantilever Blading," *Proceedings of the Institute of Mechanical Engineering*, England, Vol. 173, 1959, pp. 343-362.

[12] Subrahmanyam, K.B., Kulkarni, S.V., and Rao, J.S., "Coupled Bending Vibrations of Pre-twisted Cantilever Blading Allowing for Shear Deflection and Rotatory Inertia by the Reissner Method," *Institute of the Journal of Mechanical Sciences*, Vol. 23, No.9, 1981, pp. 517-530.

Thermal Postbuckling of Columns

G. Venkateswara Rao* and K. Kanaka Raju*
Vikram Sarabhai Space Center, Trivandrum, India

THE post-buckling behavior of columns subjected to thermal loads can be described by an eigenvalue problem. The governing matrix equation is given by

$$[K]\{\delta\} - \lambda_{NL}[G]\{\delta\} = 0 \tag{1}$$

The assembled nonlinear elastic stiffness matrix $[K]$ is obtained from the element elastic stiffness matrices which are obtained from the strain energy with nonlinear strain-displacement relations of the finite elements into which the column is discretized. $[G]$ is the assembled geometric stiffness matrix obtained from the work done by the thermal loads on each of the finite elements. Cubic polynomials in the axial coordinate are assumed for the axial and transverse displacements in evaluating the element stiffness and geometric stiffness matrices. $\{\delta\}$ is the eigenvector and λ_{NL}

Received Nov. 8, 1982; revision received July 5, 1983.

*Engineer, Aerospace Structures Division.